Free convection from a horizontal line heat source in a power-law fluid-saturated porous medium

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Free convection from a horizontal line heat source in a non-Newtonian fluid-saturated porous medium has been investigated analytically, exploiting the boundary-layer approximation. The boundary-layer equations based on the power-law model appropriate for the Darcy flows are shown to possess a class of similarity solutions for arbitrary values of the power-law index. Closed-form exact solutions for both flow and temperature fields are presented along with the approximate solutions based on the integral energy equation, and examined to investigate the non-Newtonian flow and heat transfer characteristics.

Keywords: free convection; concentrated heat source; porous media; non-Newtonian fluid

Introduction

Free convection resulting from a concentrated heat source in a porous medium is of fundamental importance in many geophysics and engineering applications, such as geophysical flows, recovery of petroleum resources, cooling of underground electric cables, and environmental impact of buried heat-generating waste. When the Rayleigh number based on the strength of heat source is sufficiently high, the classical boundary-layer theory can be exploited. Wooding (1963) developed a boundary layer treatment for plane buoyant plumes in porous media, assuming the flow to be Darcian. Wooding's initial work on the boundary-layer theory has been followed by Yih (1965), who examined the validity of the boundary-layer approximation, and pointed out that the solutions reported by Wooding for Pr = 1 are valid for all Prandtl numbers. Recently, Lai (1990) reexamined the boundary-layer analysis, and concluded that the criticism brought by Yih on the boundary-layer approximation is not valid; therefore, the boundary-layer approximation for high Rayleigh number is applicable to free convection from a concentrated heat source in a porous medium. Cheng (1978) pointed out that the problem of plume rising from a horizontal-line heat source can be considered as merely a special case of the problem of a vertical heated surface with variable well temperature. Bejan (1984) and Masuoka et al. (1986) invoked the boundary-layer approximation to obtain similarity solutions for the problem of axisymmetric plume from a point heat source in a fluid-saturated porous medium. Non-Darcy free convection from a line heat source was investigated by Lai (1991), while that from a point heat source was analyzed by Ingham (1988).

In all the above studies, the flow and temperature fields were determined under the assumption that the fluid was Newtonian. The assumption is not justified for a large class of complex fluids, such as crude oils saturating underground beds and polymer solutions in chemical engineering applications. The Darcy flow model was proposed by Christopher and Middleman (1965) for non-Newtonian power-law fluid flows in porous media, and was later modified by Dharmadhikari and Kale (1985) to improve its performance, especially for the high pseudoplastic fluids. The model was employed successfully by Chen and Chen (1988a, 1988b) to study the problem of free convection of a non-Newtonian fluid over a vertical flat plate, and later by Nakayama and Koyama (1991) to find a class of possible similarity solutions for free convection of non-Newtonian fluids over a nonisothermal body of arbitrary shape in porous media. However, no theoretical results have been reported yet for the free convection problems associated with concentrated heat sources within non-Newtonian fluid-saturated porous media.

In the present study, the Darcy flow model is used to describe the free convective flow rising from a horizontal heat source embedded in a non-Newtonian fluid-saturated porous medium. After appropriate similarity transformations based on a scale analysis, the boundary-layer equations are reduced to an ordinary differential equation. It will be shown that this ordinary differential equation possesses a remarkably simple closed form solution for an arbitrary value of the power-law index. Thus, the non-Newtonian characteristics can readily be explored.

Governing equations and scale analysis

Consider a line heat source of strength q^* embedded in a non-Newtonian fluid-saturated porous medium. Using the x and y coordinates as shown in Figure 1, the equation of continuity in terms of the apparent (Darcian) velocities can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Under the boundary-layer approximation, the non-Newtonian power laws proposed by Christopher and Middleman (1965)

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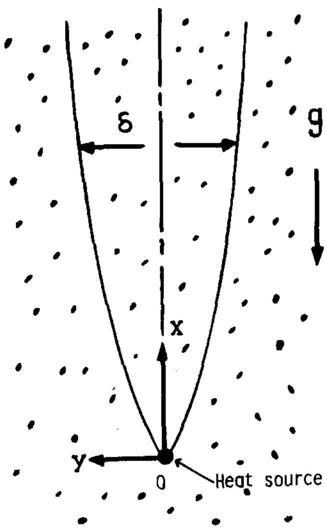

Figure 1 Physical model and its coordinates

and Dharmadhikari and Kale (1985) have a common form, which, when combined with the Boussinesq approximation, can be written as

$$\frac{\mu^*}{K^*} u^n = \rho g \beta (T - T_e) \tag{2}$$

where n is the power-law index while μ^* is the consistency index for the power-law fluid, and K^* is the modified permeability for power-law fluids as given by Nakayama and Koyama (1991) based on the studies by Christopher and Middleman (1965) and Dharmadhikari and Kale (1985). The energy equation is given as follows:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{3}$$

where α is the effective thermal diffusivity of the fluid–solid system. The energy equation must satisfy the boundary condition:

$$y = \infty: \quad T = T_e \tag{4}$$

and the enthalpy conservation constraint:

$$\rho C_p \int_{-\infty}^{\infty} u(T - T_e) dy = q^* \tag{5}$$

where ρ and C_p are the density and specific heat at constant pressure of the fluid, respectively.

A scale analysis on Equations 1, 2, 3, and 5 such as proposed by Bejan (1984), reveals that the centerline temperature and velocity decay as

$$(T_c - T_e) \sim \left\{\frac{(q^*/\rho C_p)^{2n}\mu^*}{K^* \rho g \beta (\alpha x)^n}\right\}^{1/(1+2n)} = \frac{q^*}{k_e \mathrm{Ra}_x^{1/2}} \tag{6}$$

where k_e ($= \rho C_p \alpha$) is the effective thermal conductivity of the fluid-saturated porous medium, and

$$u_c \sim \left\{\left(\frac{K^* g \beta q^*}{\mu^* C_p}\right)^2 \frac{1}{\alpha x}\right\}^{1/(1+2n)} = \frac{\alpha}{x} \mathrm{Ra}_x \tag{7}$$

The plume width δ is of the order

$$\delta \sim \left\{\frac{\mu^* C_p (\alpha x)^{1+n}}{K^* g \beta q^*}\right\}^{1/(1+2n)} = \frac{x}{\mathrm{Ra}_x^{1/2}} \tag{8}$$

where

$$\mathrm{Ra}_x = \left(\frac{K^* g \beta q^* x^n}{\mu^* C_p \alpha^{1+n}}\right)^{2/(1+2n)} \tag{9}$$

is the local Rayleigh number based on the heat source strength q^*. Thus, the boundary-layer approximation is valid when the Rayleigh number Ra_x is sufficiently large such that $\delta/x \ll 1$. Note $u_c \propto x^{-1/(1+2n)}$, $(T_c - T_e) \propto x^{-n/(1+2n)}$ and $\delta \propto x^{(1+n)/(1+2n)}$ such that $u_c \propto (T_c - T_e) \propto x^{-1/3}$ and $\delta \propto x^{2/3}$ for Newtonian fluids.

Exact solution

The preceding scale analysis prompts us to propose a set of transformations for the stream function $\psi \sim u_c \delta$ and the temperature profile function $(T - T_e) \sim (T_c - T_e)$ in terms of the similarity variable $\eta \sim y/\delta$ as follows:

$$\psi = \alpha \, \mathrm{Ra}_x^{1/2} f(\eta) \tag{10}$$

$$T - T_e = \frac{q^*}{k_e \mathrm{Ra}_x^{1/2}} \theta(\eta) \tag{11}$$

and

$$\eta = \frac{y}{x} \mathrm{Ra}_x^{1/2} \tag{12}$$

By virtue of introduction of the stream function ψ, the continuity equation (1) will be satisfied automatically. The

Notation

C_p	Specific heat of fluid at constant pressure
f	Dimensionless stream function
g	Acceleration due to gravity
k_e	Equivalent thermal conductivity of the fluid-saturated porous medium
K^*	Intrinsic permeability of the porous media for flow of power-law fluids
n	Power-law index of the inelastic non-Newtonian fluid
q^*	Strength of heat source
Ra_x	Local Rayleigh number based on q^* (defined in Equation 9)
T	Temperature
T_e	Ambient constant temperature
u, v	Darcian or superficial velocity components
x, y	Boundary-layer coordinates

Greek letters

α	Equivalent thermal diffusivity of the fluid-saturated porous medium
β	Expansion coefficient of the fluid
δ	Plume width
η	Similarity variable (defined in Equation 12)
θ	Dimensionless temperature difference
μ^*	Fluid consistency of the inelastic non-Newtonian power-law fluid
ρ	Density of the fluid
ψ	Stream function

velocity components u and v are given in terms of the proposed transformed variables as

$$u = \frac{\partial \psi}{\partial y} = \frac{\alpha}{x}\,\mathrm{Ra}_x f' \tag{13}$$

and

$$v = -\frac{\partial \psi}{\partial x} = \frac{\alpha\,\mathrm{Ra}_x^{1/2}}{x}\left(\frac{1+n}{1+2n}\eta f' - \frac{n}{1+2n} f\right) \tag{14}$$

Using Equations 11 to 14, Equations 2 and 3 can be transformed into the following set of the ordinary differential equations:

$$(f')^n = \theta \tag{15}$$

and

$$\theta'' + \frac{n}{1+2n}(f\theta)' = 0 \tag{16}$$

Integrating Equation 16 once with the condition $\theta = \theta' = 0$ as $\eta \to \infty$,

$$\theta' + \frac{n}{1+2n} f\theta = 0 \tag{17}$$

Then, combining the above equation with Equation 15 in favor of f,

$$(1+2n)f'' + ff' = 0 \tag{18}$$

It can easily be shown that the solution satisfying the foregoing equation and the boundary condition, namely, $f = 0$ at $\eta = 0$ must have the form

$$f = A \tanh\left\{\frac{A}{2(1+2n)}\eta\right\} \tag{19}$$

where constant A is determined from the enthalpy conservation constraint transformed as

$$\int_{-\infty}^{\infty} (f')^{1+n}\, d\eta = 1 \tag{20}$$

Hence, the final expressions for the streamwise velocity and temperature distributions are given by

$$\frac{u}{(\alpha\,\mathrm{Ra}_x/x)} = \frac{A^2}{2(1+2n)}\,\mathrm{sech}^2\left\{\frac{A}{2(1+2n)}\eta\right\} \tag{21}$$

and

$$\frac{T - T_e}{(q^*/k_e\,Ra_x^{1/2})} = \left\{\frac{A^2}{2(1+2n)}\right\}^n \mathrm{sech}^{2n}\left\{\frac{A}{2(1+2n)}\eta\right\} \tag{22}$$

where

$$A = \left[\frac{\{2(1+2n)\}^n}{2\int_0^\infty \mathrm{sech}^{2(1+n)} t\, dt}\right]^{1/(1+2n)} \tag{23}$$

Since $A = (\frac{9}{2})^{1/3}$ for $n = 1$, the results naturally reduce to those of Newtonian fluids, reported by Wooding (1963). The numerical values of A are furnished in Table 1.

Approximate solution

Instead of treating the energy equation in differential form, its integral form, namely, the enthalpy conservation constraint (Equation 5), is considered along with the energy balance relation along the center plane at $y = 0$ as

Table 1 Constant A

n	A
0	0.5000
0.1	0.6337
0.2	0.7640
0.3	0.8901
0.4	1.0115
0.5	1.1284
0.6	1.2408
0.7	1.3490
0.8	1.4532
0.9	1.5538
1.0	1.6510
1.1	1.7450
1.2	1.8360
1.3	1.9243
1.4	2.0101
1.5	2.0935
1.6	2.1746
1.7	2.2537
1.8	2.3308
1.9	2.4061
2.0	2.4797

$$u_c \frac{dT_c}{dx} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right)_c \tag{24}$$

where the subscript c denotes the center line. Substitution of the Darcy model (Equation 2) into Equations 5 and 24, yields

$$B\frac{\mu^* C_p}{K^* g\beta}\,\delta u_c^{1+n} = q^* \tag{25}$$

and

$$-\frac{n}{1+2n}\frac{u_c}{x} = C\alpha\frac{1}{\delta^2} \tag{26}$$

which can be combined to give

$$\delta = \left\{B\left(-C\frac{1+2n}{n}\right)^{1+n}\right\}^{1/(1+2n)} \frac{x}{\mathrm{Ra}_x^{1/2}} \tag{27}$$

where δ is an arbitrary scale for the plume width, and

$$B = \int_{-\infty}^{\infty}\left(\frac{u}{u_c}\right)^{1+n} d\left(\frac{y}{\delta}\right) \tag{28}$$

and

$$C = \frac{\partial^2}{\partial(y/\delta)^2}\left(\frac{u}{u_c}\right)^n\Bigg|_{y/\delta=0} \tag{29}$$

The shape factors B and C can readily be evaluated, since the velocity profile function is assumed as

$$u/u_c = \exp\{-(y/\delta)^2\} \tag{30}$$

such that the profile satisfies the boundary conditions required at the center plane and outside the plume. Since

$$B = \left(\frac{\pi}{1+n}\right)^{1/2}, \quad \text{and} \tag{31a}$$

$$C = -2n \tag{31b}$$

the final expression for δ is given by

$$\delta = \left\{\frac{2(1+2n)}{D}\right\}^{1/2} \frac{x}{\mathrm{Ra}_x^{1/2}} \tag{32}$$

Hence, the velocity and temperature fields can be described approximately by

$$\frac{u}{(\alpha\,\mathrm{Ra}_x/x)} = D \exp\left\{-\frac{D}{2(1+2n)}\eta^2\right\} \tag{33}$$

and

$$\frac{T-T_e}{(q^*/k_e\,\mathrm{Ra}_x^{1/2})} = D^n \exp\left\{-\frac{nD}{2(1+2n)}\eta^2\right\} \tag{34}$$

where

$$D = \left\{\frac{1+n}{2\pi(1+2n)}\right\}^{1/(1+2n)} \tag{35}$$

Results and discussion

The velocity profiles for $n = 0.5$, 1, and 2, based on exact and approximate solutions, are presented in Figure 2. The figure shows that the dilatant fluid ($n = 2$) produces a more peaked velocity profile, and its dimensionless velocity level is higher than those of the Newtonian ($n = 1$) and pseudoplastic ($n = 0.5$) fluids. The corresponding temperature profiles are presented in a similar fashion in Figure 3, where it can be seen that the dimensionless temperature level of the pseudoplastic fluid is much higher than the others, since a comparatively low velocity field prevails in the pseudoplastic fluid-saturated porous medium, as seen in the preceding figure. These figures show that both the velocity and temperature profiles from the approximate solution agree fairly well with the exact profiles. The performance of the approximate solution methodology is further examined against the exact solution in Figure 4, in terms of the velocity and temperature levels at the plume center.

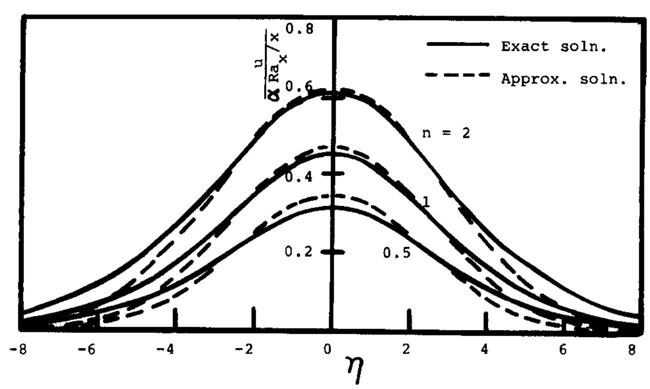

Figure 2 Velocity profiles

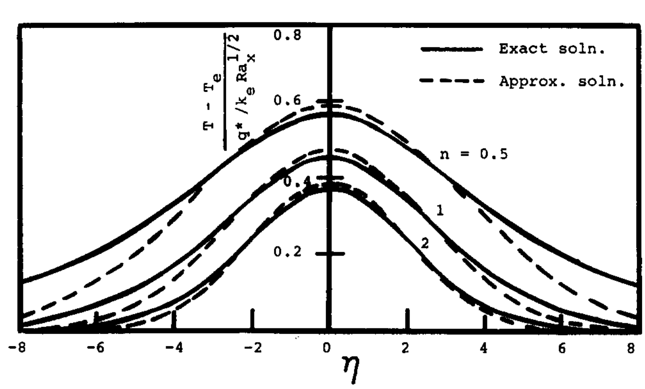

Figure 3 Temperature profiles

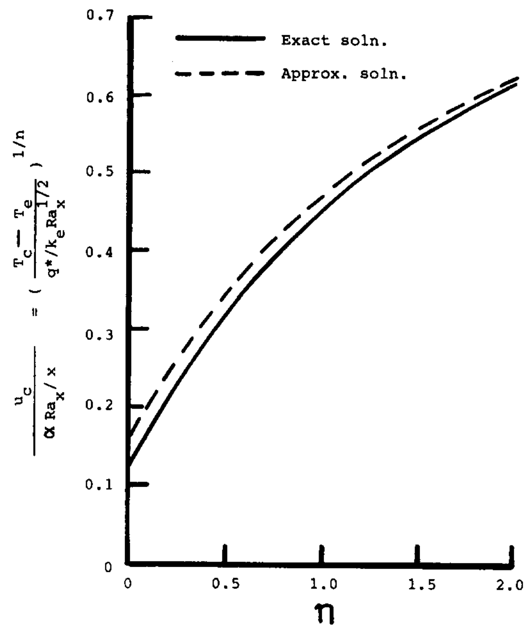

Figure 4 Effect of the power-law index on the velocity and temperature at the plume center

Although the approximate solution consistently overestimates the levels, an overall agreement between the two solutions appears to be satisfactory over the practically important range of the power-law index.

Equation 22 for the temperature field can be rewritten as

$$\frac{T-T_e}{(q^*/k_e)} = \frac{\{A^2/2(1+2n)\}^n}{\mathrm{Ra}_L^{1/2}(x/L)^{n/(1+2n)}}\,\mathrm{sech}^{2n}\left\{\frac{A\,\mathrm{Ra}_L^{1/2}}{2(1+2n)}\,\frac{(y/L)}{(x/L)^{(1+n)/(1+2n)}}\right\} \tag{36}$$

where

$$\mathrm{Ra}_L = \left(\frac{K^* g\beta q^* L^n}{\mu^* C_p \alpha^{1+n}}\right)^{2/(1+2n)} \tag{37}$$

and L is an arbitrary reference length scale. The isotherms for $(T-T_e)/(q^*/k_e) = 0.02$ generated from Equation 36 for $n = 0.5$, 1, and 2 at $\mathrm{Ra}_L = 500$ and 1,000 are presented in Figure 5, where the ordinate and abscissa scales are taken differently in

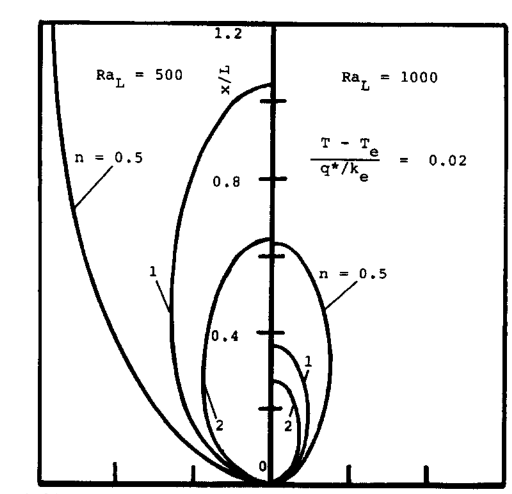

Figure 5 Isotherms for $(T-T_e)/(q^*/K_e) = 0.02$, (a) $\mathrm{Ra}_L = 500$; (b) $\mathrm{Ra}_L = 1{,}000$

order to stretch a thin plume in the y-direction. The figure shows that, for a fixed Ra_L, the warm region expands further as the power-law index decreases. It can also be seen from the figure that the temperature drops away from the heat source more drastically as Ra_L increases.

Conclusions

In this study, the plane plume resulting from a line heat source in a non-Newtonian fluid-saturated porous medium has been analyzed using the boundary-layer equations based on the power-law model. Similarity solutions are found to exist for an arbitrary value of the power-law index n. The numerical values of the constant A, which is to be determined from the energy conservation constraint, are furnished for the range of $0 \leqq n \leqq 2$. A simple integral method was also introduced to find the approximate formulas, which are found to agree fairly well with the exact solutions. The velocity and temperature fields based on the exact and approximate solutions are examined in detail to investigate the non-Newtonian fluid flow and heat transfer characteristics.

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